

ON THE ASYMPTOTICS OF THE ON-DIAGONAL SZEGÖ KERNEL OF CERTAIN REINHARDT DOMAINS

ARASH KARAMI AND VAMSI PINGALI

ABSTRACT. We compute the leading and sub-leading terms in the asymptotic expansion of the Szegő kernel on the diagonal of a class of pseudoconvex Reinhardt domains whose boundaries are endowed with a general class of smooth measures. We do so by relating it to a Bergman kernel over projective space.

1. INTRODUCTION

The Szegő and Bergman kernels play an important role in complex analysis [Gi, Be], physics [ChCo], random complex geometry [Ka, BlSh, Tr], and Kähler geometry [Ze]. It is not possible to compute them for general domains/manifolds although special cases have been considered [GaHa].

They have asymptotic expansions under very general assumptions. Computing even the first few terms of the asymptotic expansions can have many applications [Ka]. In fact we were motivated to compute them for the Szegő kernel because they appear in the study of random complex polynomials [Ka]. In this paper we compute the leading and sub-leading terms for the Szegő kernel of Reinhardt domains whose defining functions are homogeneous. Previously, similar computations were done using the Boutet De Monvel-Sjöstrand parametrix and the stationary phase method (see [Ka] and the references therein). Our computation relies on a relationship between the Bergman and Szegő kernels. While this relationship is very closely related to the well known one for unit disc bundles [Ze], the novelty lies in our allowance of more general measures on the boundary of the domain.

Acknowledgements: We thank Bernie Shiffman, George Marinescu, and Steve Zelditch for useful discussions.

2. SUMMARY OF RESULTS

In this section we review basic definitions and theorems, and present the main result of the paper.

Let $\rho^{-1}(-\infty, 1) = \Omega \subset \mathbb{C}^{n+1}$ be a domain with a smooth boundary $M = \rho^{-1}(1)$ where ρ is a smooth function on \mathbb{C}^{n+1} having non-zero gradient on M . Assume that M is endowed with a smooth volume form μ . The holomorphic tangent space T of M is defined as the complex kernel of the $(1, 0)$ form $\partial\rho$. The Hardy space $H(M, \mu)$ is defined as the space of all L^2 functions (with respect to μ) f on M such that $\bar{\partial}\tilde{f}|_T = 0$ for any extension \tilde{f} of f to a neighbourhood of M .

The Szegő kernel S of (Ω, μ) is a function $S : \Omega \times \Omega \rightarrow \mathbb{C}$ such that the orthogonal projection $\Pi : L^2(\Omega) \cap C^0(\bar{\Omega}) \rightarrow H(M, \mu)$ is given by $\Pi(f)(z) = \int_M S(z, w)f(w)\mu$. It is not hard to see that the Szegő kernel exists, is smooth away from the diagonal, and maybe computed as $S(z, w) = \sum_i \phi_i(z)\overline{\phi_i(w)}$ where ϕ_i form an orthonormal basis of $H(M, \mu)$ [Kr]. If Ω is a strictly pseudoconvex domain (i.e. the defining ρ may be chosen to be proper and strictly plurisubharmonic) then it maybe proven that the ϕ_i maybe chosen as the restriction of holomorphic functions on Ω . It is not possible to compute the Szegő kernel in general. However, in special cases one may get more information.

If ρ and μ are S^1 invariant (i.e. complete circular domains) then $H(M, \mu)$ consists of S^1 invariant functions. It may be proven the decomposition $H(M, \mu) = \oplus_{k=0}^{\infty} H_k(M, \mu)$ holds [Ka] (where the H_k are irreducible representations of S^1). Thus $S = \sum_{k=0}^{\infty} \Pi_k$. The Π_k are called partial Szegő kernels.

There exists an asymptotic expansion for it in terms of k for certain strictly pseudoconvex domains.

In this paper we restrict ourselves to domains $\Omega = \rho^{-1}[0, 1]$ where $\rho = f(|x_0|, |x_1|, \dots)$ where f is homogeneous of order l (as a consequence these domains are Reinhardt). For example, $\rho = \sum_{i=0}^2 |x_i|^4 + (|x_0 x_1|^2 + |x_0 x_2|^2 + |x_1 x_2|^2)$ on \mathbb{C}^3 . It may be proven that monomials of order k form an orthogonal basis for $H_k(M, \mu)$ with respect to the inner product in equation [Ka, Kr]. Let $J = (j_0, \dots, j_n)$, x^J be a monomial of degree n , $|J| = n$, and $x = (x_0, \dots, x_n)$. The partial Szegő kernel is

$$(2.1) \quad \Pi_k(x, x) = \sum_{|J|=n} \frac{|x^J|^2}{\langle x^J, x^J \rangle_{\mu}} \quad J = (j_0, \dots, j_n).$$

where

$$(2.2) \quad (\langle x^J, x^J \rangle_{\mu}) = \int_M |x_0|^{2j_0} \dots |x_n|^{2j_n} \mu.$$

For the sake of brevity we define

$$(2.3) \quad \psi : \mathbb{C}^{n+1} - (0, 0, \dots) \longrightarrow \mathbb{R} \quad \psi(x_0, \dots, x_n) = \rho^{-1/l}(x_0, \dots, x_n).$$

Now we define the Bergman kernel. Let (Y, ω) be an n -dimensional compact Kähler manifold and $(L^k \otimes E, h_L^k \otimes h_E)$ be a hermitian holomorphic vector bundle over Y (where L is a line bundle). The Bergman kernel B_k is defined as a section B_k of $\text{End}(E)$ such that the orthogonal projection

$$\Pi_k : L^2(Y, L^k \otimes E, \omega, h_L^k \otimes h_E) \rightarrow H(Y, L^k \otimes E, \omega, h_L^k \otimes h_E),$$

is $\Pi_k(f)(z) = \int_Y B_k(z, w) f(w) \frac{\omega^n}{n!}$ where $H(\cdot)$ is the space of holomorphic sections of $L^k \otimes E$. If (L, h_L) is ample then there is an asymptotic expansion for the Bergman kernel [MaMar, RBeBBesj]:

Theorem 2.1. *There exist smooth coefficients $b_r(y) \in \text{End}(E)_y$ which are polynomials in $\omega, -\partial\bar{\partial} \ln(h_E)$ and their derivatives and reciprocals of linear combinations of eigenvalues of $\dot{R}_{Li}^k = -(\frac{\partial^2 \log h_L}{\partial z_i \partial \bar{z}_j})(w^{k\bar{j}})$ at y , such that for any $j, l \in \mathbb{N}$, there exist $C_{k,l}$ such that for any $k \in \mathbb{N}$,*

$$(2.4) \quad |B_k(y, y) - \sum_{r=0}^l b_r(y) k^{n-r}|_{L^j(Y)} \leq C_{l,j} k^{n-l-1},$$

and

$$(2.5) \quad b_0(y) = \det \left(\frac{\dot{R}_L}{2\pi} \right) Id_E,$$

and

$$(2.6) \quad b_1 = b_0 \left(\frac{r_{\omega}}{2} - \Delta_{\omega}(\log(b_0)) + \sqrt{-1} \Lambda_{\omega}(-\partial\bar{\partial} \ln(h_E)) \right).$$

where r_{ω} is the scalar curvature of (Y, ω) in the complex geometry convention (i.e. $\omega^{i\bar{j}} \text{Ric}_{i\bar{j}}$), $\Delta_{\omega} f = -\omega^{i\bar{j}} f_{i\bar{j}}$ and $\sqrt{-1} \Lambda_{\omega} \partial\bar{\partial} \ln(h_E) = \omega^{i\bar{j}} (\ln(h_E))_{i\bar{j}}$.

For ease of notation we introduce the following matrices,

$$(2.7) \quad H(\rho) = \left(\frac{\partial^2 \rho}{\partial x_i \partial \bar{x}_j} \right)_{0 \leq i, j \leq n} \text{ and } H_0(\rho) = \left(\frac{\partial^2 \rho}{\partial x_i \partial \bar{x}_j} \right)_{1 \leq i, j \leq n},$$

$$(2.8) \quad \nabla \rho(x) = \left(\frac{\partial \rho}{\partial \bar{x}_0}, \dots, \frac{\partial \rho}{\partial \bar{x}_n} \right) \text{ and } \nabla_0 \rho(x) = \left(\frac{\partial \rho}{\partial \bar{x}_1}, \dots, \frac{\partial \rho}{\partial \bar{x}_n} \right).$$

Finally we state our main theorem

Theorem 2.2. *If $x = (x_0, \dots, x_n) \in \Omega$ where $\Omega = \rho^{-1}[0, 1)$ is defined earlier, then the first two terms of the asymptotic expansion of the reproducing kernel of the projection map*

$$(2.9) \quad \Pi_k : L^2(\Omega, \mu) \longrightarrow H_k(M),$$

are

$$a_0(x, x) = \left(\frac{2}{l} \right)^{n+2} \frac{\det(H(\rho))}{2\pi^{n+1} e^{u(x\psi(x))} \psi^{2n-l(n+1)} \sqrt{\left(\frac{\partial \psi}{\partial |x_0|} \right)^2 + \left(\frac{\partial \psi}{\partial |x_1|} \right)^2 + \dots}},$$

and

$$a_1(x, x) = \frac{a_0}{4} \left(2n(n+1) + 2 \sum_{\mu=0}^n |x|^2 \frac{\partial^2 \ln(a_0)}{\partial x_\mu \partial \bar{x}_\mu} \right),$$

where $\Pi_k(x, x) = a_0(x, x)k^{n+1} + a_1(x, x)k^n + O(k^{n-1}) + \dots$

3. PROOF OF THE MAIN THEOREM

In order to compute the asymptotic expansion of S , we relate the partial Szegő kernel Π_k of Ω with the Bergman kernel B_k of $(\mathcal{O}(k) \otimes E, h_k \otimes h_E)$ over $(\mathbb{CP}^n, \omega_{FS})$ where E is a trivial line bundle. The Fubini-study metric in a chart is $\omega_{FS} = i\partial\bar{\partial} \ln(1 + |z|^2)$. After doing so we use the asymptotic expansion of the Bergman kernel as given in [MaMar], [RBeBBesj].

In order to simplify computations define

$$(3.1) \quad \psi : \mathbb{C}^{n+1} - (0, 0, \dots) \longrightarrow \mathbb{R} \quad \psi(x_0, \dots, x_n) = \rho^{-1/l}(x_0, \dots, x_n).$$

The gradient of ρ is nonzero on $M = \rho^{-1}(1)$ and hence M is a $2n + 1$ -dimensional manifold admitting a natural S^1 action : $x \rightarrow e^{i\theta}x$. Identifying $\tilde{M} = M/S^1$, we have a quotient map

$$\tilde{p} : M \longrightarrow \tilde{M} = M/S^1 \quad \tilde{\pi}(x) = \tilde{x}.$$

Lemma 3.1. *\tilde{M} is an n -dimensional complex manifold biholomorphic to \mathbb{CP}^n via a map $\pi : \mathbb{CP}^n \rightarrow \tilde{M}$.*

Proof. Write $\tilde{M} = \cup_{i=0}^n \tilde{M}_i$ where $\tilde{M}_i = \{(x_0, \dots, x_n) \in \tilde{M} : x_i \neq 0\}$ and define

$$\phi_0 : \mathbb{C}^n \longrightarrow \tilde{M}_0 \quad \phi_0(w_1, \dots, w_n) = \psi(1, w_1, \dots, w_n)(1, w_1, \dots, w_n),$$

and similarly we define ϕ_i . The ϕ_i are certainly homeomorphisms. To show that $\phi_i^{-1} \circ \phi_j$ is holomorphic, we do so for $i = 1, j = 0$ and the same proof works for different i, j .

(3.2)

$$\begin{aligned} \phi_1^{-1} \circ \phi_0(w_1, \dots, w_n) &= \phi_1^{-1}(\psi(1, w_1, \dots, w_n)(1, w_1, \dots, w_n)) \\ &= \phi_1^{-1}(\psi(1, 1/w_1, \dots, w_n/w_1)(1/w_1, 1, \dots, w_n/w_1)) = (1/w_1, \dots, w_n/w_1). \end{aligned}$$

Hence $\phi_1^{-1} \circ \phi_0$ is holomorphic on $\phi_0^{-1}(\tilde{M}_0 \cap \tilde{M}_1)$.

Define the map

$$\pi : \mathbb{CP}^n \longrightarrow M \text{ where}$$

$$\pi([x_0, \dots, x_n]) = \psi(x_0, \dots, x_n)(x_0, \dots, x_n).$$

It is easy to see that $\tilde{p} \circ \pi$ is a well-defined biholomorphism onto \tilde{M} . \square

Lemma 3.2. *The defining function ρ induces a Hermitian metric h on the hyperplane section bundle $(O(1), \mathbb{CP}^n)$ such that*

$$(3.3) \quad \begin{aligned} h_\beta([z_0, z_1, \dots, 1, z_{\beta+1}, \dots]) &= \psi^2(z_0, z_1, \dots, 1, z_{\beta+1}, \dots) \text{ on } U_\beta \text{ and,} \\ \|e_\alpha\|_{h^k}^2 &= \pi^*|x^\alpha|^2. \end{aligned}$$

Proof. The collection of h_β does patch up to give a Hermitian metric on $O(1)$. Indeed, the transition functions $g_{\beta\gamma}$ for $O(1)$ on $U_\beta \cap U_\gamma$ are equal to $\frac{x_\beta}{x_\gamma}$, $g_{\beta\gamma}([x_0, \dots, x_n]) = \frac{x_\beta}{x_\gamma}$ and by definition of h_β we have

$$\frac{h_\beta}{h_\gamma} = \frac{|x_\beta|^2}{|x_\gamma|^2} = |g_{\beta\gamma}|^2.$$

Moreover on U_β ,

$$(3.4) \quad \begin{aligned} \|e_\alpha\|_{h^k}^2 &= |z_0^{\alpha_0} z_1^{\alpha_1} \dots z_{\beta-1}^{\alpha_{\beta-1}} z_{\beta+1}^{\alpha_{\beta+1}} \dots|^2 h_\beta^k \\ &= |z_0^{\alpha_0} z_1^{\alpha_1} \dots z_{\beta-1}^{\alpha_{\beta-1}} z_{\beta+1}^{\alpha_{\beta+1}} \dots|^2 \psi^{2k}(z_0, z_1, \dots, 1, z_{\beta+1}, \dots) \\ &= \pi^*|x^\alpha|^2. \end{aligned}$$

\square

Let μ_{ind} be the volume form on M induced from the Euclidean metric on \mathbb{C}^{n+1} . An arbitrary S^1 -invariant volume form μ on M may be written as $\mu = e^u \mu_{ind}$ where u is a smooth function on \tilde{M} . We want to find a function h_E on \mathbb{CP}^n such that $\int_{\mathbb{CP}^n} \pi^*(\mathcal{F}) h_E \frac{\omega_{FS}^n}{n!} = \int_M \mathcal{F} \mu$ for every S^1 -invariant function \mathcal{F} on M . To this end, notice that

$$(3.5) \quad \begin{aligned} \pi^* dx_0 &= d\psi(1, z_1, \dots, z_n) \text{ and} \\ \pi^* dx_i &= d(z_i \psi(1, z_1, \dots, z_n)) = z_i d\psi + \psi dz_i \text{ when } i=1, \dots, n. \end{aligned}$$

Lemma 3.3. *There is a smooth function h_E on \mathbb{CP}^n satisfying*

$$h_E([y_0, y_1, \dots, y_n]) = 2\pi \pi^*(e^u) |y|^{2n+2} \left(\frac{\psi^2(|y_0|, |y_1|, \dots)}{2} \right)^n \sqrt{\left(\frac{\partial \psi}{\partial |y_0|} \right)^2 + \left(\frac{\partial \psi}{\partial |y_1|} \right)^2 + \dots},$$

and

$$\int_{\mathbb{CP}^n} \pi^*(\mathcal{F}) h_E \frac{\omega_{FS}^n}{n!} = \int_M \mathcal{F} \mu,$$

for every S^1 -invariant function \mathcal{F} on M .

Proof. We denote coordinates in \mathbb{C}^{n+1} by (x_0, x_1, \dots) , and homogeneous coordinates on \mathbb{CP}^n by $[y_0, \dots]$. Define $z_i = \frac{y_i}{y_0}$ when $y_0 \neq 0$ and let $x_i = R_i e^{\sqrt{-1}\Theta_i}$. Let us assume that R_0 may be solved

for as a function $R_0 = f(R_1, \dots, R_n)$ on a domain D . The metric on M induced from the Euclidean one on \mathbb{C}^{n+1} is

$$\begin{aligned} g &= df \otimes df + f^2 d\Theta_0^2 + \sum_{i=1}^n dR_i^2 + R_i^2 d\Theta_i^2 \\ (3.6) \quad &= \frac{\partial f}{\partial R_i} \frac{\partial f}{\partial R_j} dR_i dR_j + f^2 d\Theta_0^2 + \sum_{i=1}^n dR_i^2 + R_i^2 d\Theta_i^2. \end{aligned}$$

It is easy to see that the corresponding volume form is

$$\mu_{ind} = f \sqrt{1 + \sum_{i=1}^n \left(\frac{\partial f}{\partial R_i} \right)^2} d\Theta_0 R_1 dR_1 d\Theta_1 R_2 dR_2 d\Theta_2 \dots$$

We may write this expression more invariantly as follows :

$$\begin{aligned} \psi(f, R_1, R_2, \dots) &= 1 \\ \frac{\partial \psi}{\partial R_0} \frac{\partial f}{\partial R_i} + \frac{\partial \psi}{\partial R_i} &= 0 \\ (3.7) \quad \mu_{ind} &= R_0 \frac{\sqrt{\sum_{i=0}^n \left(\frac{\partial \psi}{\partial R_i} \right)^2}}{\left| \frac{\partial \psi}{\partial R_0} \right|} d\Theta_0 R_1 dR_1 d\Theta_1 R_2 dR_2 d\Theta_2 \dots \end{aligned}$$

Thus the integral over M of a continuous S^1 -invariant function \mathcal{F} is easily seen to be

$$(3.8) \quad 2\pi \int_D \mathcal{F} R_0 \frac{\sqrt{\sum_{i=0}^n \left(\frac{\partial \psi}{\partial R_i} \right)^2}}{\left| \frac{\partial \psi}{\partial R_0} \right|} R_1 dR_1 d\Theta_1 R_2 dR_2 d\Theta_2 \dots$$

Our task is reduced to calculating $\pi^*(R_1 dR_1 d\Theta_1 R_2 dR_2 d\Theta_2 \dots)$ on U_0 which is the same as $\pi^* \frac{\tilde{\omega}^n}{n!}$ where $\tilde{\omega} = \frac{\sqrt{-1}}{2} \sum_{i=1}^n dx_i \wedge d\bar{x}_i$ on \mathbb{C}^{n+1} . For $z = [1, z_1, \dots, z_n] \in U_0$ we have

$$\begin{aligned} \pi^* \tilde{\omega} &= \frac{\sqrt{-1}}{2} \sum_{i=1}^n \pi^* dx_i \wedge \pi^* d\bar{x}_i \\ &= \frac{\sqrt{-1}}{2} \sum_{i=1}^n (z_i d\psi + \psi dz_i) \wedge (\overline{z_i d\psi + \psi dz_i}) \\ (3.9) \quad &= \frac{\sqrt{-1}}{2} \sum_{i=1}^n z_i \bar{\psi} d\psi \wedge d\bar{z}_i + \frac{\sqrt{-1}}{2} \sum_{i=1}^n \bar{z}_i \psi dz_i \wedge d\bar{\psi} + \psi^2 \frac{\sqrt{-1}}{2} \sum_{i=1}^n dz_i \wedge d\bar{z}_i \\ &= \frac{\sqrt{-1}}{2} \psi \sum_{i=1}^n d\psi \wedge (z_i d\bar{z}_i - \bar{z}_i dz_i) + \frac{\sqrt{-1}}{2} \psi^2 \sum_{i=1}^n dz_i \wedge d\bar{z}_i. \end{aligned}$$

If we use cylindrical polar coordinates $z_j = r_j e^{\sqrt{-1}\theta_j}$, then

$$(3.10) \quad z_i d\bar{z}_i - \bar{z}_i dz_i = -2\sqrt{-1} r_i^2 d\theta_i.$$

We then compute $d\psi$:

$$(3.11) \quad d\psi = \sum_{j=1}^n \frac{\partial \psi}{\partial r_j} dr_j.$$

So in cylindrical polar coordinates we have

$$(3.12) \quad \begin{aligned} \pi^* \tilde{\omega} &= \psi \sum_{i=1}^n \sum_{j=1}^n \left(\frac{\partial \psi}{\partial r_j} r_i \right) dr_j \wedge d\theta_i + \psi^2 \sum_{i=1}^n r_i dr_i \wedge d\theta_i \\ &= \sum_{i=1}^n \sum_{j=1}^n (r_i \frac{\partial \psi}{\partial r_j} \psi + \delta_{ij} \psi^2) r_i dr_j \wedge d\theta_i, \end{aligned}$$

thus implying that

$$(3.13) \quad \begin{aligned} \pi^* \frac{\tilde{\omega}^n}{n!} &= \left(\psi^{2n} + \psi^{2n-1} \sum_{i=1}^n r_i \frac{\partial \psi}{\partial r_i} \right) r_1 dr_1 d\theta_1 r_2 dr_2 d\theta_2 \dots \\ &= \left(-2^n \psi^{2n-1} \frac{\partial \psi(|y_0|, |y_1|, \dots)}{\partial |y_0|} \Big|_{(1, r_1, \dots)} \right) (1 + |z|^2)^{n+1} \frac{\omega_{FS}^n}{n!}, \end{aligned}$$

where the last equality follows from the homogeneity of ψ . We want to choose h_E such that

$$(3.14) \quad \int_{\mathbb{CP}^n} \pi^* \mathcal{F} h_E \frac{\omega_{FS}^n}{n!} = \int_{\mathbb{C}^n} \pi^* (2\pi \int_D \mathcal{F} R_0 \frac{\sqrt{\sum_{i=0}^n \left(\frac{\partial \psi}{\partial R_i} \right)^2}}{\left| \frac{\partial \psi}{\partial R_0} \right|} R_1 dR_1 d\Theta_1 R_2 dR_2 d\Theta_2 \dots).$$

Writing equation 3.13 in terms of y_0, y_1, \dots and substituting in the above expression we see that

$$(3.15) \quad h_E([y_0, y_1, \dots, y_n]) = 2\pi \pi^*(e^u) |y|^{2n+2} \left(\frac{\psi^2(|y_0|, |y_1|, \dots)}{2} \right)^n \sqrt{\left(\frac{\partial \psi}{\partial |y_0|} \right)^2 + \left(\frac{\partial \psi}{\partial |y_1|} \right)^2 + \dots}$$

□

The function h_E is smooth on \mathbb{CP}^n and so it may be seen as a Hermitian metric on the trivial bundle (E, \mathbb{CP}^n) . Finally we have the following useful relationship,

Lemma 3.4. *The Bergman kernel B_k of $(\mathbb{CP}^n, \omega_{FS}, \mathcal{O}(k) \otimes E, h_k \otimes h_E)$ restricted to $\mathbb{CP}^n \times \mathbb{CP}^n$ is related to the partial Szego kernel Π_k of (M, μ) restricted to $M \times M$ as*

$$(3.16) \quad \pi^* \Pi_k(x, x) = \frac{B_k([x], [x])}{h_E}.$$

Proof: The Bergman kernel is

$$\begin{aligned} B_k([x], [x]) &= \sum_{|\alpha|=k} \frac{\|e_\alpha\|_{h_k \otimes h_E}^2}{\int_{\mathbb{CP}^n} \|e_\alpha\|_{h_k \otimes h_E}^2 \frac{\omega_{FS}^n}{n!}} \\ &= \sum_{|\alpha|=k} \frac{\pi^*(|x^\alpha|^2) h_E}{\langle x^\alpha, x^\alpha \rangle_\mu} \\ &= \pi^* \Pi_k(x, x) h_E. \end{aligned}$$

3.1. Curvature of the Hermitian metrics. Here we compute the curvature of the Hermitian metric h . For the sake of brevity we denote

$$(3.17) \quad \rho_1 = \frac{\partial \rho}{\partial z_1}, \dots, \rho_n = \frac{\partial \rho}{\partial z_n}, \rho_{i\bar{j}} = \frac{\partial^2 \rho}{\partial z_i \partial \bar{z}_j},$$

$$(3.18) \quad A = (\rho \rho_{i\bar{j}})_{1 \leq i, j \leq n}, \text{ and a vector } v = (\bar{\rho}_1, \dots, \bar{\rho}_n).$$

Therefore the curvature of h at the point $[1, z_1, \dots, z_n] \in U_0$ is

$$(3.19) \quad \begin{aligned} \Theta_h &= -\partial \bar{\partial} \ln h = -2\partial \bar{\partial}(\psi(1, z_1, \dots, z_n)) \\ &= \frac{2}{l} \partial \bar{\partial} \ln \rho(1, z_1, \dots, z_n) = \frac{2}{l} \sum \left(\frac{\rho \rho_{i\bar{j}} - \rho_i \rho_{\bar{j}}}{\rho^2} \right) dz_i \wedge d\bar{z}_j. \end{aligned}$$

The preceding expression for Θ_h may be used to prove its positivity. Indeed,

Lemma 3.5. *The curvature of the Hermitian metric h on the hyperplane section bundle $(O(1), \mathbb{CP}^n)$ is positive.*

Proof. We use the plurisubharmonicity of the defining function ρ in the following form:

$$(3.20) \quad \partial_x \bar{\partial}_x \rho((a, \vec{w}), (\bar{a}, \bar{\vec{w}})) > 0,$$

for every $a \in \mathbb{C}$ and $\vec{w} \in \mathbb{C}^n$. Notice that

$$\begin{aligned} \partial_x \bar{\partial}_x \rho &= \frac{\partial^2}{\partial x_0 \partial \bar{x}_0} (|x_0|^l \rho(1, z_1, \dots)) dx_0 \wedge d\bar{x}_0 + \frac{\partial^2}{\partial x_0 \partial \bar{z}_i} (|x_0|^l \rho(1, z_1, \dots)) dx_0 \wedge d\bar{z}_i \\ &\quad + \frac{\partial^2}{\partial z_i \partial \bar{x}_0} (|x_0|^l \rho(1, z_1, \dots)) dz_i \wedge d\bar{x}_0 + |x_0|^l \partial_z \bar{\partial}_z \rho. \\ \partial_x \bar{\partial}_x \rho((a, \vec{w}), (\bar{a}, \bar{\vec{w}})) &= |x_0|^{l-2} \left| al + \frac{x_0}{\rho} \sum_i \frac{\partial \rho}{\partial z_i} w_i \right|^2 + |x_0|^l \partial_z \bar{\partial}_z \rho(\vec{w}, \bar{\vec{w}}) - \frac{|x_0|^l}{\rho} \left| \sum_i \frac{\partial \rho}{\partial z_i} w_i \right|^2. \end{aligned}$$

Choosing $a = \frac{-1}{l} \sum \frac{X_0}{\rho} \frac{\partial \rho}{\partial z_i} w_i$, and using expressions 3.19 and 3.20, we see that the curvature Θ_h is positive. \square

The following computation is useful :

Lemma 3.6. *The determinant of the curvature matrix on the coordinate chart U_0 is*

$$\det(\Theta_h) = \left(\frac{2}{l\rho^2} \right)^n \det(A)(1 - vA^{-1}v^*).$$

Proof. Recall that

$$(3.21) \quad \frac{l}{2} (\rho^2 \Theta_h)_{j\bar{j}} = (\rho \rho_{i\bar{j}} - \rho_i \rho_{\bar{j}}).$$

If we let A_i be the i -th column of the matrix A then

$$(3.22) \quad \frac{l}{2} \rho^2 \Theta_h = (A_1 - \rho_1 v^*, \dots, A_n - \rho_n v^*),$$

Hence

$$\begin{aligned}
 (3.23) \quad \frac{l^n \rho^{2n}}{2^n} \det(\Theta_h) &= \det(A) - \rho_1 \det(v^*, A_2, \dots, A_n) - \dots - \rho_n \det(A_1, \dots, v^*) \\
 &= \det(A) \left(1 - \sum_{i=1}^n \rho_i \frac{\det(\tilde{A}_i)}{\det(A)}\right) \text{Cramer's rule,} \\
 &= \det(A) (1 - vA^{-1}v^*).
 \end{aligned}$$

Therefore

$$(3.24) \quad \det(\Theta_h) = \left(\frac{2}{l\rho^2}\right)^n \det(A) (1 - vA^{-1}v^*).$$

□

It is easy to see that

Proposition 3.7. *If $f(|x_0|, |x_1|, \dots)$ is a homogeneous function of order l ,*

$$\begin{aligned}
 \frac{|x_0|^l}{x_0} f_i &= \frac{\partial f}{\partial x_i}, \\
 \frac{|x_0|^l}{\bar{x}_0} f_{\bar{i}} &= \frac{\partial f}{\partial \bar{x}_i}, \\
 |x_0|^{l-2} f_{i\bar{j}} &= \frac{\partial^2 f}{\partial x_i \partial \bar{x}_j}.
 \end{aligned}$$

We then rewrite equation (3.24) in the x-coordinates :

Lemma 3.8. *The determinant of the curvature matrix on U_0 of the Hermitian metric h in the x-coordinates is*

$$(3.25) \quad \det(\Theta_h) = \left(\frac{2}{l}\right)^{n+2} \left(\frac{|x_0|^2}{\rho}\right)^{n+1} \det(H(\rho)).$$

Proof. Using proposition 3.7 we may write A and v in the x-coordinates. That is,

$$(3.26) \quad A = \rho(\rho_{i\bar{j}}) = \frac{\rho(x)}{|x_0|^{2l-2}} H_0(\rho),$$

and

$$(3.27) \quad v = \frac{1}{|x_0|^{l-1}} \nabla_0 \rho.$$

Then

$$vA^{-1}v^* = \frac{1}{\rho(x)} (\nabla_0 \rho) H_0^{-1}(\rho) (\nabla_0 \rho)^*.$$

So we have

$$\begin{aligned}
\det(\Theta_h) &= \left(\frac{2}{l\rho^2}\right)^n \det(A)(1 - vA^{-1}v^*) \\
&= \left(2\frac{|x_0|^{2l}}{l\rho^2(x)}\right)^n \left(\frac{\rho(x)}{|x_0|^{2l-2}}\right)^n \det(H_0(\rho))(1 - \frac{1}{\rho(x)}(\nabla_0\rho)H_0^{-1}(\rho)(\nabla_0\rho)^*) \\
(3.28) \quad &= \left(\frac{2|x_0|^2}{l\rho(x)}\right)^n \det(H_0(\rho))(1 - \frac{1}{\rho(x)}(\nabla_0\rho)H_0^{-1}(\rho)(\nabla_0\rho)^*) \\
&= \left(\frac{2|x_0|^2}{l\rho}\right)^n \frac{1}{\rho} \det(H_0(\rho))(\rho - (\nabla_0\rho)H_0^{-1}(\rho)(\nabla_0\rho)^*).
\end{aligned}$$

We then proceed to write $\det(H(\rho))$ in terms of $H_0(\rho)$. To do this we use the homogeneity of ρ in the form

$$\begin{aligned}
\sum_{i=1}^n x_i \frac{\partial \rho}{\partial x_i} + x_0 \frac{\partial \rho}{\partial x_0} &= \frac{l\rho}{2}, \\
(3.29) \quad \sum_{i=1}^n \bar{x}_i \frac{\partial \rho}{\partial \bar{x}_i} + \bar{x}_0 \frac{\partial \rho}{\partial \bar{x}_0} &= \frac{l\rho}{2}.
\end{aligned}$$

Differentiating the first equation in 3.29 with respect to \bar{x}_j we see that

$$\begin{aligned}
\sum_{i=1}^n x_i \frac{\partial^2 \rho}{\partial x_i \partial \bar{x}_j} + x_0 \frac{\partial^2 \rho}{\partial x_0 \partial \bar{x}_j} &= \frac{l}{2} \frac{\partial \rho}{\partial \bar{x}_j}, \\
(3.30) \quad \frac{\partial^2 \rho}{\partial x_0 \partial \bar{x}_0} &= \frac{l^2 \rho}{4|x_0|^2} - \frac{l}{2|x_0|^2} \sum_{i=1}^n \bar{x}_i \frac{\partial \rho}{\partial \bar{x}_i} - \frac{x_i}{x_0} \frac{\partial^2 \rho}{\partial x_i \partial \bar{x}_0}.
\end{aligned}$$

So by using the equation 3.30 we have:

$$(3.31) \quad \frac{\partial^2 \rho}{\partial x_0 \partial \bar{x}_j} = \frac{l}{2x_0} \frac{\partial \rho}{\partial \bar{x}_j} - \sum_{i=1}^n \frac{x_i}{x_0} \frac{\partial^2 \rho}{\partial x_i \partial \bar{x}_j}.$$

The equations 3.30, 3.31 imply that the determinant of the matrix $H(\rho)$ is (using row operations)

$$\begin{aligned}
(3.32) \quad &\left| \begin{array}{ccc} \left(\frac{l^2 \rho}{4|x_0|^2} - \frac{l}{2|x_0|^2} \sum_{i=1}^n \bar{x}_i \frac{\partial \rho}{\partial \bar{x}_i} - \sum_{i=1}^n \frac{x_i}{x_0} \frac{\partial^2 \rho}{\partial x_i \partial \bar{x}_0}\right) & \cdots & \left(\frac{l}{2x_0} \frac{\partial \rho}{\partial \bar{x}_n} - \sum_{i=1}^n \frac{x_i}{x_0} \frac{\partial^2 \rho}{\partial x_i \partial \bar{x}_n}\right) \\ \vdots & & \\ \left(\frac{\partial^2 \rho}{\partial x_n \partial \bar{x}_0}\right) & & H_0 \end{array} \right| \\
&= \left| \begin{array}{cc} \left(\frac{l^2 \rho}{4|x_0|^2} - \frac{l}{2|x_0|^2} \sum_{i=1}^n \bar{x}_i \frac{\partial \rho}{\partial \bar{x}_i}\right) & \cdots \frac{l}{2x_0} \frac{\partial \rho}{\partial \bar{x}_n} \\ \vdots & \\ \frac{l}{2x_0} \frac{\partial \rho}{\partial \bar{x}_n} - \sum_{i=1}^n \frac{x_i}{x_0} \frac{\partial^2 \rho}{\partial x_i \partial \bar{x}_n} & H_0 \end{array} \right| = \frac{l}{2x_0} \left| \begin{array}{cc} \left(\frac{l\rho}{2x_0} - \frac{1}{x_0} \sum_{i=1}^n \bar{x}_i \frac{\partial \rho}{\partial \bar{x}_i}\right) & \cdots \frac{\partial \rho}{\partial \bar{x}_n} \\ \vdots & \\ \frac{l}{2x_0} \frac{\partial \rho}{\partial \bar{x}_n} - \sum_{i=1}^n \frac{x_i}{x_0} \frac{\partial^2 \rho}{\partial x_i \partial \bar{x}_n} & H_0 \end{array} \right| \\
&= \frac{l}{4|x_0|^2} \left| \begin{array}{cc} \rho & \cdots \frac{\partial \rho}{\partial \bar{x}_n} \\ \vdots & \\ \frac{\partial \rho}{\partial \bar{x}_n} & H_0 \end{array} \right|.
\end{aligned}$$

At this point we split the determinant as

$$\frac{l}{4|x_0|^2} \det(A + B) = \frac{l}{4|x_0|^2} \det(A) \det(I + A^{-1}B),$$

where the matrices A and B are

$$A = \begin{bmatrix} \rho & 0 \\ 0 & H_0 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & \cdots & \frac{\partial \rho}{\partial x_n} \\ \vdots & & \\ \frac{\partial \rho}{\partial x_n} & & 0_{n \times n} \end{bmatrix}.$$

Hence the determinant equals $\frac{l^2}{4|x_0|^2} \det(H_0) \det(I + A^{-1}B)$ which is easily evaluated to be

$$\frac{l^2}{4|x_0|^2} \det(H_0) (\rho - (\nabla_0 \rho) H_0^{-1} (\rho) (\nabla_0 \rho)^*).$$

This in conjunction with equation 3.28 proves the lemma. \square

3.2. Terms of the asymptotic expansion. Finally we may prove theorem 2.2 which is stated once again for the reader's convenience :

Theorem 3.9. *If $x = (x_0, \dots, x_n) \in \Omega$ where $\Omega = \rho^{-1}[0, 1)$ is defined earlier, then the first two terms of the asymptotic expansion of the reproducing kernel of the projection map*

$$(3.33) \quad \Pi_k : L^2(\Omega, \mu) \longrightarrow H_k(M),$$

are

$$a_0(x, x) = \left(\frac{2}{l}\right)^{n+2} \frac{\det(H(\rho))}{2\pi^{n+1} e^{u(x\psi(x))} \psi^{2n-l(n+1)} \sqrt{\left(\frac{\partial \psi}{\partial |x_0|}\right)^2 + \left(\frac{\partial \psi}{\partial |x_1|}\right)^2 + \dots}},$$

and

$$a_1(x, x) = \frac{a_0}{4} \left(2n(n+1) + 2 \sum_{\mu=0}^n |x|^2 \frac{\partial^2 \ln(a_0)}{\partial x_\mu \partial \bar{x}_\mu} \right),$$

where $\Pi_k(x, x) = a_0(x, x)k^{n+1} + a_1(x, x)k^n + O(k^{n-1}) + \dots$

Proof. By using equation lemma 3.4, theorem 2.1, lemma 3.3, and lemma 3.8 we have

$$(3.34) \quad a_0(x, x) = \frac{1}{h_E} \frac{1}{(2\pi)^n} \frac{\det(\Theta_h)}{\det(\Theta_{FS})}$$

$$(3.35) \quad = \frac{1}{h_E} \frac{1}{(2\pi)^n} \frac{\left(\frac{2}{l}\right)^{n+2} \left(\frac{|x_0|^2}{\rho}\right)^{n+1} \det(H(\rho))}{\frac{|x_0|^{2n+2}}{(|x_0|^2 + \dots + |x_n|^2)^{n+1}}}$$

$$(3.36) \quad = \frac{1}{h_E} \frac{1}{(2\pi)^n} \left(\frac{2}{l}\right)^{n+2} \left(\frac{|x|^2}{\rho}\right)^{n+1} \det(H(\rho))$$

$$(3.37) \quad = \left(\frac{2}{l}\right)^{n+2} \frac{\det(H(\rho))}{2\pi^{n+1} e^{u(x\psi(x))} \psi^{2n-l(n+1)} |d\psi|}$$

An easy application of lemma 3.4, theorem 2.1, lemma 3.3, and proposition 3.7 shows that

$$\begin{aligned} a_1(x, x) &= \frac{a_0}{4}(2n(n+1) + 4 \left(\sum_{i=1}^n |x|^2 \frac{\partial^2 \ln(a_0 h)}{\partial x_i \partial \bar{x}_i} + \frac{|x|^2}{|x_0|^2} \bar{x}_i x_j \frac{\partial^2 \ln(a_0 h)}{\partial x_i \partial \bar{x}_j} \right) \\ &\quad - 4 \left(\sum_{i=1}^n |x|^2 \frac{\partial^2 \ln(h)}{\partial x_i \partial \bar{x}_i} + \frac{|x|^2}{|x_0|^2} \bar{x}_i x_j \frac{\partial^2 \ln(h)}{\partial x_i \partial \bar{x}_j} \right)). \end{aligned}$$

Using equations 3.30 with $l = 0$ yields the desired formula for a_1 . Hence we have

$$a_1(x, x) = \frac{a_0}{4} \left(2n(n+1) + 2 \sum_{\mu=0}^n |x|^2 \frac{\partial^2 \ln(a_0)}{\partial x_\mu \partial \bar{x}_\mu} \right).$$

□

REFERENCES

- [Be] S.R.Bell. The Szego kernel and proper holomorphic mappings to a half plane. *Comput. Methods Funct. Theory* 11 (2011), no. 1, 179191.
- [BlSh] T.Bloom, B.Shiffman. Zeros Of random Polynomials on \mathbb{C}^m . *Math. Res. Lett* 14(2007), 469-479.
- [Ca] C.Carracino. Estimates for the Szegő kernel on a model non-pseudoconvex domain. *Illinois J. Math.* 51 (2007), no. 4, 13631396.
- [ChCo] J.S.R. Chisholm, A.K. Common. Clifford Algebras and Their Applications in Mathematical Physics . *Springer*, Jul 31, 1986 - Mathematics - 616 pages
- [GaHa] F.Gabor, N.Hanges. Explicit formula for the Szegő kernel and certain weakly psuedoconvex domains. *Proceedings of the American mathematical society* 1995, no.10.
- [Gi] M.A.Gilliam. The Szego Kernel for Non-Pseudoconvex Domains in \mathbb{C}^2 . *Thesis (Ph.D.)University of Montana*. 2011. 106 pp. ISBN: 978-1124-71443-1, ProQuest LLC
- [Ka] A.Karami. Zeros of random Reinhardt polynomials. *arxiv* 1207.5764,
- [Kr] S.Krantz, *Function theory of several complex variables* (2ed., AMS, 1992)(ISBN 0534170889)
- [MaMar] X.Ma, G.Marinescu. Holomorphic Morse inequalities and Bergman kernels. Vol. 254. *Birkhauser Verlag Basel*, 2007.
- [Tr] T.Tran. Continuity of the asymptotics of expected zeros of fewnomials. *arxiv* 1311.7168,
- [Ze] S.Zelditch. Szego kernels and a theorem of Tian. *Internat. Math. Res. Notices* 1998, no. 6, 317331.
- [RBeBBesj] R.Berman, B.Berndtsson, J.Sjöstrand. A direct approach to Bergman kernel asymptotics for positive line bundles. *Arkiv fr matematik* (2008) 46(2), 197-217.

DEPARTMENT OF MATHEMATICS, JOHNS HOPKINS UNIVERSITY, BALTIMORE, MD 21210, USA
E-mail address: akarami@math.jhu.edu

KRIEGER 412, DEPARTMENT OF MATHEMATICS, JOHNS HOPKINS UNIVERSITY, BALTIMORE, MD 21210, USA
E-mail address: vpingali@math.jhu.edu